ISRAEL JOURNAL OF MATHEMATICS, Vol. 51, No. 3, 1985

AN ERGODIC THEOREM ON BANACH LATTICES

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ABSTRACT

E is a Banach lattice that is weakly sequentially complete and has a weak unit u. TL $f_n = \phi$ means that the infimum of $|f_n - \phi|$ and u converges strongly to zero. T is a positive contraction operator on E and $A_n = (1/n)(I + T + \cdots + T^{n-1})$. Without an additional assumption on E, the "truncated limit" TL $A_n f$ need not exist for f in E. This limit exists for each f if E satisfies the following additional assumption (C): For every f in E_+ and for every number $\alpha > 0$, there is a number $\beta = \beta(f, \alpha)$ such that if g is in E_+, $\|g\| \leqq 1$, $0 \leqq f' \leqq f$ and $\|f'\| > \alpha$ then $\|f' + g\| \geqq \|g\| + \beta$.

We consider Banach lattices E that are weakly sequentially complete (Condition (B) below) and have a weak unit u (Condition (A)), i.e., an element $u \in E_+$ such that $|f| \wedge u = 0$ implies $f = 0$. If f_n are positive, $\phi \in E_+$ is called the (weak) truncated limit of f_n, if for each positive integer k, $f_n \wedge ku$ converges (weakly) to ϕ_k and $\phi_k \uparrow \phi$. We then write $\phi = \text{(W)TL} f_n$. There is sequential compactness for WTL, which can be used in ergodic theory instead of Banach limits and other non-constructive arguments. Weak truncated limits were (implicitly) introduced in [1] and applied to superadditive ergodic theory. They were studied in [3], in the context of Banach lattices. A related notion was considered by Brooks and Chacon [4]; see also Ghoussoub and Steele [11]. If $E = L_1$ then any norm bounded positive sequence has a subsequence that decomposes into g_n and h_n in L_1^+ such that g_n converges weakly, say to ϕ, and the h_n's have disjoint supports. Then the limit function ϕ has the desired properties of a weak truncated limit. However, in Banach lattices that we consider this decomposition may fail; see [3] for a discussion.

† Research of this author is partially supported by NSERC Grant A3974.

†† Research of this author is partially supported by NSF Grant 8301619.

Received May 29, 1984 and in revised form November 7, 1984

In a recent paper [3] we applied truncated limits to the problem of existence of positive elements invariant under positive operators. Here we study the limiting behaviour of averages

$$A_n = \frac{1}{n}(I + T + \cdots + T^{n-1})$$

of a positive contraction T. If E is L_1 of a probability space then $A_n f$, $f \in L_1$, does not converge in L_1, or, as shown by Chacon [6], almost everywhere. However, Krengel [13] proved that $A_n f$ converges in probability; see also [2]. In L_1, f_n converges in probability to ϕ if and only if TL $f_n = \phi$ [2]. This raises the natural question whether there are ergodic theorems for TL convergence in Banach lattices. We show here by an example that the conditions (A) and (B) are not sufficient. Additional assumptions are needed ensuring that if a positive function increases so does its norm. One such assumption is (C_1): $\|f + g\| > \|g\|$ whenever $f, g \in E_+$ and $f \neq 0$. Under (C_1) we prove that if $0 \leqq \phi = T\phi$ then the (strong) limit of $(A_n f) \wedge \phi$ exists. The main theorem asserting the existence of the (strong) truncated limit of $A_n f$, $f \in E_+$, is proved under the following condition (C), a "uniform" version of (C_1).

(C) For every $f \in E_+$ and for every number $\alpha > 0$ there is a number $\beta = \beta(f, \alpha) > 0$ such that if $g \in E_+$, $\|g\| \leqq 1$, $0 \leqq f' \leqq f$ and $\|f'\| > \alpha$ then $\|f' + g\| \geqq \|g\| + \beta$.

It is an open problem whether (C_1) is sufficient for the existence of TL $A_n f$.

The paper is self-contained except for some basic facts about Banach lattices, for which we refer to [15] and for the classical mean ergodic theorem. In the first section we give the needed elements of the theory of truncated limits. The ergodic theorem is proved in the second section.

1. Properties of truncated limits

Let E be a Banach lattice. Our terminology will be that of the book Lindenstrauss–Tzafriri [15], to which we will refer by [LT]. First we will make only the following two assumptions (A) and (B):

(A) There is an element u in E_+ such that if f is in E_+ and if $u \wedge f = 0$, then $f = 0$. Such an element u is called a *weak unit*.

(B) Every norm-bounded increasing sequence in E has a strong limit.

Assumptions equivalent with (B) are: (B′) E is weakly sequentially complete, and also: (B″) E contains no isomorphic copy of c_0 ([LT], p. 34). (B) implies that E is order-continuous. Therefore, the assumption (A) that there is a weak unit is not a loss of generality if E is separable ([LT], p. 9).

Since the condition (B) implies order-continuity, one has

1.1. Every order interval $[f, g] = \{h : t \leqq h \leqq g\}$ is weakly compact ([LT], p. 28).

Norm convergence will be simply called *convergence* and denoted by $\rightarrow$. Weak convergence is $\xrightarrow{w}$, and order convergence for monotone sequences is denoted by $\uparrow$ and $\downarrow$.

1.2. Let $\phi \in E_+$. Then there is a linear bounded operator $P = P_\phi : E \rightarrow E$ such that $Pf = \lim f \wedge (n\phi)$ for each $f \in E_+$ (limit in *strong* topology). Then P is a *band projection* (on ϕ), implying that $\|P\| \leqq 1$, $P^2 = P$ and PE is a sub Banach lattice of E. $Q = 1 - P$ is another band projection.

1.3. A band projection P_u on a weak unit u is the identity, i.e., if $f \in E_+$, then $f \wedge nu \rightarrow f$.

In other terms, a weak unit is necessarily a *quasi-interior point* ([17], p. 96) or a *topological unit.*

1.4. There exists a *strictly positive* element U in E^*_+, i.e., a U such that $f = 0$ if $U|f| = 0$ ([LT], p. 25; if E is separable, this is very easy to prove).

1.5. If f_n in E_+ is such that $f_n \xrightarrow{w} 0$ and $\sup f_n \in E$, then $f_n \rightarrow 0$.

Proof. If the conclusion fails, then passing to subsequences we can ssume that $\|f_n\| > \varepsilon > 0$ for all n and $\Sigma\, Uf_n < \infty$ for a strictly positive U in E^*_+. Let $g_n = \bigvee_{k=n}^{\infty} f_k$, $g = \bigwedge_{n=1}^{\infty} g_n$. Then $g_n \downarrow g$ implies that $g_n \rightarrow g$, hence $Ug_n \downarrow Ug$. But $Ug \leqq \Sigma_{k \geqq n} Uf_k$ implies that $Ug = 0$, hence $g = 0$. This contradicts $\|g\| = \lim \|g_n\| > \varepsilon$.

Definition of truncated limits

Let $f_n \in E_+$, $\phi \in E_+$. Then $\mathrm{TL}\, f_n = \phi$ (the *truncated limit* of f_n is ϕ) means that for a weak unit u, $\lim_n (f_n \wedge ku) = \phi_k$ exists for each k, and $\phi_k \uparrow \phi$.

For f_n in E, $\mathrm{TL}\, f_n = \mathrm{TL}\, f_n^+ - \mathrm{TL}\, f_n^-$, provided the truncated limits to the right exist. This definition is independent of the choice of the weak unit u (cf. 1.3).

We define analogously the WTL's (weak truncated limits), requiring only that $f_n \wedge ku \xrightarrow{w} \phi_k$.

TL null sequences

A sequence (f_n) is called TL null if $\mathrm{TL}\,|f_n|=0$. For this it suffices that $|f_n|\wedge u \xrightarrow{n} 0$ (cf. 1.5). If E is L^1 of a measure space, then, as shown in [2], TL null sequences are exactly the sequences of functions that converge to zero in measure on sets of finite measure.

Since the main theorem asserts TL convergence, it is of interest to observe that also in more general Banach lattices TL convergence implies convergence in measure on sets of finite measure. This follows from the following lemma, together with Theorem 1.b.14 [LT], which reduces the discussion to the L^1 case.

1.6. LEMMA. *Let $f_n \in E$, $\phi \in E$. Then* $\mathrm{TL}\,f_n = \phi$ *if and only if $g_n = f_n - \phi$ is* TL-*null.*

PROOF. We at first suppose that $f_n \in E^+$, $\phi \in E^+$. Assume that g_n is TL-null.

$$|f_n \wedge ku - \phi \wedge ku| \leqq |f_n - \phi| \wedge ku \xrightarrow{n} 0.$$

Hence $f_n \wedge ku \xrightarrow{n} \phi_k = \phi \wedge ku \uparrow \phi$. Thus $\mathrm{TL}\,f_n = \phi$. Conversely, if g_n is not TL null, then there is a $v \in E^+$ such that $\limsup_n \||g_n|\wedge v\| = \alpha > 0$. Find k_0 sufficiently large such that $\|v-(v\wedge k_0u)\| < \alpha/2$. Then $v = k_0u + v'$ with $\|v'\| < \alpha/2$. Hence $|g_n|\wedge v \leqq |g_n|\wedge k_0u + v'$ implies

$$\limsup_n \|g_n \wedge k_0 u\| \geqq \alpha/2 \qquad \text{for each } k \geqq k_0.$$

We will show that this is a contradiction if $\mathrm{TL}\,f_n = \phi$. In fact, choose $k \geqq k_0$ such that $\|\phi - \phi_k\| < \alpha/8$ and $\|\phi - \phi\wedge ku\| < \alpha/8$. Then

$$|f_n - \phi| \wedge ku \leqq |f_n \wedge 2ku - \phi \wedge ku| + |\phi - \phi\wedge ku|$$

implies that

$$\limsup_n \||f_n - \phi|\wedge ku\| \leqq \|\phi_{2k} - \phi\wedge ku\| + \|\phi - \phi \wedge ku\| < 3\frac{\alpha}{8} < \frac{\alpha}{2}.$$

Thus $\mathrm{TL}\,|f_n - \phi| = 0$. Now consider f_n and ϕ in E without assuming positivity. By definition, $\mathrm{TL}\,f_n = \phi$ if $\mathrm{TL}\,f_n^+ = \phi^+$ and $\mathrm{TL}\,f_n^- = \phi^-$, and the converse is also true. Therefore it suffices to show that $f_n - \phi$ is TL-null if and only if both $f_n^+ - \phi^+$ and $f_n^- - \phi^-$ are TL-null. This follows from

$$|f_n^+ - \phi^+| \vee |f_n^- - \phi^-| \leqq |f_n - \phi| \leqq |f_n^+ - \phi^+| + |f_n^- - \phi^-|.$$

1.7. LEMMA. *If $f_n \geqq 0$,* $\mathrm{WTL}\,f_n = \phi$. *Let $P = P_\phi$ be the band projection on ϕ and let $Q = Q_\phi = I - P$* (cf. 1.2). *Then* $\mathrm{WTL}\,(Pf_n) = \phi$ *and Qf_n is* TL-*null.*

PROOF. Let $f_n \wedge (ku) \xrightarrow{w} \phi_k$, $\phi_k \uparrow \phi$. Since $P\phi_k = \phi_k$, we also have that $P(f_n \wedge ku) \xrightarrow{w} \phi_k$. Hence the intermediate sequence $(Pf_n) \wedge ku$ also converges weakly to ϕ_k, and therefore $(Qf_n) \wedge ku \xrightarrow{w} 0$.

The most useful result of this section is the sequential compactness for WTL. It suffices to state it for positive sequences.

1.8. PROPOSITION. *If $f_n \geqq 0$ and $\sup \|f_n\| = M < \infty$, then there is a subsequence (f_{n_i}) such that* WTL $f_{n_i} = \phi$ *exists. If f_n is not a* TL-*null sequence, then this subsequence can be chosen so that $\phi \neq 0$.*

PROOF. Apply 1.1 to intervals $[0, ku]$ for $k = 1, 2, \ldots$. The sequence f_{n_i} obtained by diagonal procedure will be such that $f_{n_i} \wedge ku \xrightarrow{w} \phi_k$ for each k, and $\phi_k \leqq \phi_{k+1}$. Since $\|\phi_k\| \leqq M$, $\phi = \lim \uparrow \phi_k \in E$.

Now if $\|f_n \wedge u\| \not\to 0$, then passing to a subsequence we can ssume that $\|f_n \wedge u\| > \alpha > 0$ for all n. Then no subsequence $f_{n_i} \wedge u$ can converge weakly to zero, because by 1.5 it would converge strongly to zero.

1.9. PROPOSITION. *Let $f_n, g_n \in E_+$,* WTL $f_n = \phi$, WTL $g_n = \gamma$.

(a) *If* WTL $(f_n + g_n) = \psi$ *exists then $\psi = \phi + \gamma$.*

(b) *If $T: E \to E$ is a positive linear operator and $Tf_n = g_n$ then $T\phi \leqq \gamma$.*

PROOF. (a) Since for each k one has

$$(f_n + g_n) \wedge ku \leqq (f_n \wedge ku) + (g_n \wedge ku),$$

the inequality $\psi \leqq \phi + \gamma$ is clear. In the opposite direction use the inequality

$$(f_n + g_n) \wedge 2ku \geqq (f_n \wedge ku) + (g_n \wedge ku).$$

Let $f_n \wedge ku \xrightarrow{w} \phi_k$, $g_n \wedge ku \xrightarrow{w} \gamma_k$ and $(f_n + g_n) \wedge ku \xrightarrow{w} \psi_k$. Then the last inequality implies that $\psi_{2k} \geqq \phi_k + \gamma_k$. On letting $k \to \infty$, we have $\psi \geqq \phi + \gamma$.

(b) Let ϕ_k and γ_k be as before. Given k and $\epsilon > 0$, find m so large that

$$\|Tku - (Tku) \wedge (mu)\| < \epsilon.$$

Then

$$\begin{aligned} T\phi_k &= T[\text{weak lim}\,(f_n \wedge ku)] \\ &= \text{weak lim}\, T(f_n \wedge ku) \\ &\leqq \text{weak lim}\, Tf_n \wedge Tku \end{aligned}$$

implies that

$$T\phi_k \leqq \text{weak}\lim Tf_n \wedge mu + r$$

$$= \gamma_m + r,$$

with $\|r\| < \epsilon$. Since ϵ is arbitrary, it follows that $T\phi \leqq \gamma$. □

Finally we need a simple result involving strong limits of truncated sequences. If $\phi \in E_+$ then we write

$$\mathrm{TL}_\phi f_n = \lambda$$

to mean that $\lim_n f_n \wedge k\phi = \lambda_k$ exists for each k and that $\lambda_k \uparrow \lambda \in E_+$. Let P_ϕ be the band projection on ϕ.

1.10. LEMMA. $\mathrm{TL}_\phi f_n = \lambda$ *if and only if* $\mathrm{TL}\, P_\phi f_n = \lambda$.

PROOF. First observe that if $\lim_n g_n = g$ in E_+ then $\lim_n (g_n \wedge v) = g \wedge v$ for any $v \in E_+$ (continuity of lattice operations in strong topology). To prove the "only if" part, assume that $f_n \wedge k\phi \to \lambda_k$ and $\lambda_k \uparrow \lambda$. Then $(P_\phi f_n) \wedge v$ converges strongly to $\lambda \wedge v$ for any $v \in E_+$. In fact, given $\epsilon > 0$ we can find k_0 such that $\|\lambda - \lambda_{k_0}\| < \epsilon$ and such that $\|v \wedge (k_0\phi) - P_\phi v\| < \epsilon$. Then

$$P_\phi f_n \wedge v = f_n \wedge P_\phi v = f_n \wedge (v \wedge k_0\phi) + \nu_n \qquad \text{where } 0 \leqq \nu_n \leqq P_\phi v - v \wedge k_0\phi$$

and hence $\|\nu_n\| < \epsilon$. But $f_n \wedge (v \wedge k_0\phi) = (f_n \wedge k_0\phi) \wedge v \to \lambda_{k_0} \wedge v$. Therefore

$$\limsup_n \|P_\phi f_n \wedge v - \lambda \wedge v\| \leqq \limsup_n [\|f_n \wedge (v \wedge k_0\phi) - \lambda \wedge v\| + \|\nu_n\|]$$

$$\leqq \|\lambda_{k_0} \wedge v - \lambda \wedge v\| + \epsilon \leqq 2\epsilon.$$

The proof of the "if" part is similar. □

2. The ergodic theorem for TL convergence

As in Section 1, we assume that E is a Banach lattice satisfying (A) and (B). Let $T : E \to E$ be a linear positive operator and let

$$A_n(T) = A_n = \frac{1}{n}(1 + T + \cdots + T^{n-1})$$

be the Cesaro averages of the iterates of T. We investigate the existence of $\mathrm{TL}(A_n f)$ for $f \in E$. We assume that T is a contraction, i.e., $\|T\| \leqq 1$.

If E is L_1 of a probability space, $f \in E_+$, then $A_n f$ need not converge almost everywhere or in norm, but $A_n f$ converges in probability (Krengel [13]). In L^1 the convergence in probability is exactly the strong TL convergence, but the following simple example shows that the strong truncated limit of $A_n f$ need not exist without an additional assumption on the lattice E.

EXAMPLE. E is the space l_1 of absolutely summable sequences with the following different norm: $f = (f_0; f_1, f_2, \dots) \in E$, then

$$\|f\| = \max\left(f_0, \sum_{i=1}^{\infty} |f_i|\right).$$

If $(\alpha_1, \alpha_2, \dots)$ is any sequence of real numbers with $0 \leqq \alpha_i \leqq 1$, then T defined as

$$T(f_0; f_1, f_2, \dots) = \left(\sum_{i=1}^{\infty} \alpha_i f_i ; 0, f_1, f_2, \dots\right)$$

is a positive linear contraction on E. Let $f = (0; 1, 0, 0, \dots)$, then

$$T^n f = (\alpha_n, \underbrace{0, 0, \dots, 0}_{n \text{ times}}, 1, 0, 0 \dots),$$

$$A_n f = \left(\frac{1}{n}\sum_{i=1}^{n-1} \alpha_i, \frac{1}{n}, \dots, \frac{1}{n}, 0, 0, \dots\right),$$

and if the Cesaro averages of the α_i's diverge, then the truncated limit of $A_n f$ clearly does not exist. The lattice E satisfies (A) and (B), but what is wrong with E is that if, e.g., $f = (1; 0, 0, 0, \dots)$ and $g = (0; 1, 0, 0, \dots)$, then $\|f + g\| = \|f\| = \|g\| > 0$.

The following lemma guarantees the existence of subinvariant functions for T. For a more general version of this lemma see Lemma 2.2 in [3].

2.1. LEMMA. *Let $f \in E_+$. If for a sequence (n_i),* WTL $A_{n_i} f = \phi$ *then $T\phi \leqq \phi$. If $A_n f$ is not a* TL*-null sequence then there is a sequence (n_i) such that* WTL $A_{n_i} f = \phi \neq 0$.

PROOF. Since $\|A_{n_i} f - TA_{n_i} f\| \to 0$ as $i \to \infty$, we see that WTL $TA_{n_i} f = \phi$. Then Proposition 1.9 shows that $T\phi \leqq \phi$. If $A_n f$ is not TL-null then, by Proposition 1.8, there is a subsequence (n_i) such that WTL $A_{n_i} f = \phi \neq 0$.

2.2. LEMMA. *Let $\|T\| \leqq 1$. If $\phi \in E_+$, $T\phi = \phi$, $f \in E_+$, $f \leqq \phi$ then $A_n f$ converges strongly.*

PROOF. This easily follows from the Kakutani–Yosida mean ergodic theorem (cf. [8], VIII 5.1 or [16], p. 442) since $A_n f \leqq \phi$ implies that $(A_n f)$ is weakly compact by (1.1). □

To proceed we need assumptions guaranteeing that if f, g are non-zero positive elements then $\|f+g\| > \|g\|$. This will eliminate examples of the type discussed above. The following assumption (C_1) allows one to prove strong convergence of $A_n f \wedge \phi$, if $T\phi = \phi$ (Theorem 2.4). The main result, the existence of a strong truncated limit of $A_n f$ (Theorem 2.8), is proved under the stronger assumption (C).

(C_1) For every $f, g \in E_+$, if $\|f\| \neq 0$ then $\|f+g\| > \|g\|$.

2.3. LEMMA. *Assume* (C_1). *Given $\phi \in E_+$ with $T\phi = \phi$ and a number $\alpha > 0$, there is a number $\sigma = \sigma(\phi, \alpha) > 0$ such that if $0 \leqq f \leqq \phi$ and $\|f\| \geqq \alpha$ then* $\lim \|A_n f\| \geqq \sigma$.

PROOF. If for a $g \in E_+$, $\lim A_n \phi$ exists, denote this limit by $\bar{g}$. If the lemma is not true, then there is an invariant $\phi \in E_+$, an $\alpha > 0$, and elements f_n in E_+ with $f_n \leqq \phi$, $\|f_n\| > \alpha$, $\|\bar{f}_n\| \to 0$.

Passing to a subsequence, we may assume that $\|\bar{f}_n\| \leqq \epsilon_n$, $\Sigma \epsilon_n < \infty$. Let $g_n = \bigvee_{k=n}^{\infty} f_k$, $g = \lim \downarrow g_n$. Then $\|g_n\| \geqq \alpha$, hence $\|g\| \geqq \alpha$, but $\|\bar{g}_n\| \leqq \Sigma_n^{\infty} \epsilon \to 0$, hence $\|\bar{g}\| = 0$. Now $\phi = A_n \phi = A_n g + A_n(\phi - g) \to \overline{(\phi - g)}$. It follows that $\|\phi\| = \|\overline{(\phi - g)}\| \leqq \|\phi - g\|$. Since $0 \leqq g \leqq \phi$, (C_1) implies that $g = 0$, which is a contradiction. □

2.4. THEOREM. *Let E satisfy* (A), (B) *and* (C_1), $\|T\| \leqq 1$, $\phi \in E_+$, $T\phi = \phi$, $f \in E_+$. *Then $(A_n f) \wedge \phi$ converges strongly.*

PROOF. Let $g_k = \phi \wedge A_k f$. Then for a fixed k, $A_n g_k$ is eventually dominated by g_n in the sense that

$$\lim_n \|A_n g_k - (A_n g_k) \wedge g_n\| = 0. \tag{2.4.1}$$

In fact, $g_k \leqq A_k f$ and $g_k \leqq \phi$ implies that $A_n g_k \leqq A_n \phi = \phi$ and $A_n g_k \leqq (A_n A_k f) \wedge \phi$. But $\|A_n A_k f - A_n f\| \to 0$ for a fixed k, and consequently, this gives (2.4.1). Since by Lemma 2.2 $\bar{g}_k = \lim A_n g_k$ exists, we also have

$$\|\bar{g}_k - \bar{g}_k \wedge g_n\| \to 0 \qquad (n \to \infty). \tag{2.4.2}$$

Now define for each k

$$\alpha_k = \limsup_{n \to \infty} \|g_n - (g_n \wedge \bar{g}_k)\|.$$

Then we have $\alpha_k \to 0$ as $k \to \infty$. Indeed, supposing that $\limsup \alpha_k > \alpha > 0$ we will obtain a contradiction.

We first show that given any k and n_0 and $\epsilon > 0$ we can find $n > n_0$ and two functions $p, q \in E_+$, $p \leqq \phi$, $q \leqq \phi$ such that $\|p\| > \alpha$, $\|q\| < \epsilon$, and

$$g_n = \bar{g}_k + p - q. \tag{2.4.3}$$

In fact we only have to take

$$p = g_n - (g_n \wedge \bar{g}_k),$$

$$q = \bar{g}_k - (g_n \wedge \bar{g}_k)$$

for a sufficiently large integer n. Since $0 \leqq p \leqq \phi$ and $0 \leqq q \leqq \phi$, we also have that $\bar{p} = \text{strong}\lim_{n\to\infty} A_n p$, $\bar{q} = \text{strong}\lim_{n\to\infty} A_n q$ exist and of course $\|\bar{q}\| < \epsilon$.

By Lemma 2.3 we have $\|\bar{p}\| > \sigma$, where $\sigma > 0$ is a constant that depends only on ϕ and α. Hence from (2.4.3) we have

$$\bar{g}_n = \bar{g}_k + \bar{p} - \bar{q}$$

for some $n > n_0$, $\|\bar{p}\| > \sigma$, $\|\bar{q}\| < \epsilon$. This gives a contradiction as follows: let $\epsilon_i > 0$, $\sum_{i=1}^{\infty} \epsilon_i = \epsilon < \infty$. Assuming $n_0, n_1, \ldots, n_{i-1}$ already chosen, choose $n_i > n_{i-1}$ such that

$$\bar{g}_n = \bar{g}_{n_{i-1}} + \bar{p}_i - \bar{q}_i$$

where $\|\bar{p}_i\| > \sigma$, $\|\bar{q}_i\| < \epsilon_i$. Hence

$$\|\bar{g}_{n_i} - \bar{g}_{n_{i-1}}\| = \|\bar{p}_i - \bar{q}_i\| > \sigma - \epsilon_i \not\to 0.$$

This is the desired contradiction as we will now show that $\bar{g}_{n_i}$ must converge strongly. In fact, since

$$\bar{g}_{n_1} \leqq \bar{g}_{n_2} + \bar{q}_2 \leqq \bar{g}_{n_3} + \bar{q}_2 + \bar{q}_3 \leqq \cdots$$

we see, by induction, that $\bar{g}_{n_i} + \sum_{j=2}^{i} \bar{q}_j$ is an increasing sequence. But $\bar{g}_{n_i} \leqq \phi$ and $\|\sum_{j=2}^{i} \bar{q}_j\| \leqq \sum_{j=1}^{\infty} \epsilon_j < \infty$, which means that this sequence is also norm bounded. Hence it converges strongly. Since $\sum_{j=2}^{i} \bar{q}_j$ is also strongly convergent, this shows that $\bar{g}_{n_i}$ is strongly convergent.

Consequently we now know that

$$\limsup_n \|g_n - (g_n \wedge \bar{g}_k)\| = \alpha_k \to 0 \qquad \text{as } k \to \infty.$$

To continue the proof, let $\epsilon > 0$ be given. Find k_0 such that $\alpha_{k_0} < \epsilon$. Now find n_0 such that $n \geqq n_0$ implies that

$$\|g_n - (g_n \wedge \bar{g}_{k_0})\| < \epsilon$$

and

$$\|\bar{g}_{k_0}-(g_n \wedge \bar{g}_{k_0})\|<\epsilon.$$

Hence if $n \geqq n_0$ then $g_n = \bar{g}_{k_0} + r_n - s_n$ with $\|r_n\|<\epsilon$, $\|s_n\|<\epsilon$, where

$$r_n = g_n - (g_n \wedge \bar{q}_{k_0}),$$

$$s_n = \bar{g}_{k_0} - (g_n \wedge \bar{g}_{k_0}).$$

Therefore, if $n, m > n_0$, then

$$\|g_n - g_m\| = \|r_n - r_m - s_n + s_m\| < 4\epsilon.$$

Hence g_n converges strongly. □

2.5. LEMMA. *If* $f_n \in E_+$, $\sup_n \|f_n\| = M < \infty$, $\mathrm{WTL}\, f_n = \phi$, $f_n \wedge k\phi \xrightarrow{w} \lambda_k$, $\lambda_k \uparrow \lambda$, *then* $\lambda = \phi$.

PROOF. Since u is a unit, $\lambda \leqq \phi$. In the opposite direction, let $f_n \wedge ku \xrightarrow{w} \phi_k \uparrow \phi$. Given $\epsilon > 0$, choose k such that $\|\phi - \phi_k\| < \epsilon$, and then choose m such that

$$\|P(ku) - ku \wedge m\phi\| < \epsilon,$$

where P is the band projection on ϕ. Hence for *all* integers n,

$$\|P(f_n \wedge ku) - (f_n \wedge ku) \wedge m\phi\| < \epsilon$$

and the expression inside $\|\ \ \|$ is positive. Therefore replacing in it $(f_n \wedge ku) \wedge m\phi$ by the bigger element $f_n \wedge m\phi$, we obtain that for all n

$$\|(P(f_n \wedge ku) - f_n \wedge m\phi)^+\| < \epsilon.$$

Now let $n \to \infty$; it follows that $\|(\phi_n - \lambda_m)^+\| < \epsilon$, and $\|(\phi - \lambda_m)^+\| < 2\epsilon$. Therefore $\|(\phi - \lambda)^+\| < 2\epsilon$, $\phi \leqq \lambda$. □

From Lemma 2.1 we know that for a sequence (n_i), $\mathrm{WTL}\,(A_{n_i} f) = \phi$ exists and $T\phi \leqq \phi$. We need, however, $T\phi = \phi$. This will be implied by the following assumption (C) on the Banach lattice E.

(C) For every F in E_+ and for every number $\alpha > 0$ there is a number $\beta = \beta(f, \alpha) > 0$ such that if $g \in E_+$, $\|g\| \leqq 1$, $0 \leqq f' \leqq f$ and $\|f'\| \geqq \alpha$ then $\|g + f'\| \geqq \|g\| + \beta$.

2.6. LEMMA. *Assume* (A), (B) *and* (C). *If* $\phi = \mathrm{WTL}\, A_{n_i} f$, $f \in E_+$, *then* $T\phi = \phi$.

PROOF. Suppose, without loss of generality, $\|f\| \leqq 1$. Let $\phi' = \phi - T\phi$ and assume that $\phi' \neq 0$. Replacing (n_i) by a further subsequence we may assume that $\text{weak}\lim_{i\to\infty} A_{n_i}f \wedge m\phi = \psi_m$ exists for each $m = 1, 2, \ldots$. By Lemma 2.5, $\psi_m \uparrow \phi$. Fix m so that $\|\phi - \psi_m\| < \frac{1}{4}\|\phi'\|$. Let

$$r_i = A_{n_i}f \wedge m\phi, \qquad s_i = A_{n_i}f - r_i,$$

$$r'_i = (TA_{n_i}f) \wedge m\phi, \qquad s'_i = TA_{n_i}f - r'_i.$$

Since $\|A_{n_i}f - TA_{n_i}f\| \to 0$, we have both

$$\|r_i - r'_i\| \to 0 \quad \text{and} \quad \|s_i - s'_i\| \to 0. \tag{2.6.1}$$

Now $TA_{n_i}f = Tr_i + Ts_i = r'_i + s'_i$. Since $T\phi \leqq \phi$, we have that $Tr_i \leqq m\phi$. This means that $0 \leqq Tr_i \leqq r'_i$ and $Ts_i = s'_i + (r'_i - Tr_i)$ with both summands positive. We have, by (2.6.1), that $r'_i \xrightarrow{w} \psi_m$. Since $r_i \xrightarrow{w} \psi_m$, and hence $Tr_i \xrightarrow{w} T\psi_m$, we have that $r'_i - Tr_i \xrightarrow{w} \psi_m - T\psi_m$. But

$$\begin{aligned}\psi_m - T\psi_m &= \phi - T\phi + (\psi_m - \phi) - T(\psi_m - \phi)\\ &= \phi' + (\psi_m - \phi) - T(\psi_m - \phi),\end{aligned}$$

which implies that

$$\|\psi_m - T\psi_m\| \geqq \|\phi'\| - 2\|\psi_m - \phi\| \geqq \tfrac{1}{2}\|\phi'\|.$$

Therefore $\liminf_{i\to\infty}\|r'_i - Tr_i\| > \frac{1}{4}\|\phi'\|$. Let $\beta = \beta(m\phi, \frac{1}{4}\|\phi'\|)$, as given in (C). Then we see that $\|Ts_i\| = \|s'_i + (r'_i - Tr_i)\| \geqq \|s'_i\| + \beta$ for all sufficiently large i. This is a contradiction, since $\|T\| \leqq 1$ and since $\|s_i - s'_i\| \to 0$. Hence $\phi' = 0$. □

2.7. LEMMA. *Assume* (A), (B) *and* (C). *If* (n_i) *and* (m_j) *are two sequences such that* $\text{WTL}\, A_{n_i}f = \phi$ *and* $\text{WTL}\, A_{m_j}f = \psi$ *then* $\phi = \psi$.

PROOF. We know that $A_nf \wedge k\phi$ converges strongly, say, to ϕ_k. Then, by Lemma 2.5, applied to the sequence (n_i), we see that $\phi_k \uparrow \phi$.

It is again convenient to use the notation TL_ϕ, defined before Lemma 1.10. We have $\text{TL}_\phi A_nf = \phi$. Hence, by Lemma 1.10, this implies $\text{TL}\, P_\phi A_nf = \phi$. Then $\text{WTL}\, P_\phi A_nf = \phi$ and hence $\text{WTL}\, P_\phi A_{m_j}f = \phi$. But also $\text{WTL}\, A_{m_j}f = \psi$. Since $P_\phi A_{m_j}f \leqq A_{m_j}f$ we see that $\phi \leqq \psi$. By symmetry, $\psi \leqq \phi$. □

We now state the main theorem of the paper.

2.8. THEOREM. *Assume that E satisfies* (A), (B) *and* (C). *Let* $f \in E_+$. *Then the strong truncated limit* $\text{TL}\, A_nf = \phi$ *exists and* $T\phi = \phi$.

PROOF. Let $\phi = \text{WTL}\, A_{n_i}f$ for a subsequence (n_i). We know that

$\mathrm{TL}\, P_\phi A_n f = \phi$ exists. By compactness of $A_n f$ for WTL and Lemma 2.7 we know that every subsequence $A_n f$ has a further subsequence with weak truncated limit equal to ϕ. Let $Q = 1 - P_\phi$. Then every subsequence of $QA_n f$ contains a further subsequence with strong truncated limit equal to zero (Lemma 1.7). We show that this implies $\mathrm{TL}\, QA_n f = 0$. Otherwise there is a subsequence (m_j) such that $\|(QA_{m_j} f) \wedge u\| > \epsilon > 0$. Then no further subsequence of $QA_{m_j} f$ can have strong truncated limit zero. Since $A_n f = P_\phi A_n f + QA_n f$ and $\mathrm{TL}\, P_\phi A_n f = \phi$ and $\mathrm{TL}\, QA_n f = 0$, we see easily that $\mathrm{TL}\, A_n f = \phi$. □

2.9. EXAMPLE. In the case of an Orlicz space $L_M = L_M(X, \mathcal{F}, \mu)$, where μ is a finite non-atomic measure, the condition (C) is equivalent to the condition Δ_2. Although this is rather straightforward to see, we give a complete proof, using the notation and some of the basic results in the book *Convex Functions and Orlicz Spaces* (Groningen, 1961) by Krasnosel'skii and Rutickii. Let $M(u) = \int_0^u p(t)dt$, $u \geqq 0$, where $p(t)$ is defined for $t \geqq 0$, right continuous and non-decreasing, such that $p(0) = 0$, $p(t) > 0$ if $t > 0$ and $p(t) \to \infty$ as $t \to \infty$. Note that

$$M(u + v) \geqq M(u) + M(v) \qquad \text{for all } u, v \geqq 0.$$

The set L_M consists of all functions $f : X \to \mathbb{R}$ such that

$$\int M\left(\frac{1}{s}|f|\right) < \infty \qquad \text{for some number } s > 0.$$

This becomes a Banach space with the norm

$$\|f\| = \inf\left\{ s \mid s > 0, \int M\left(\frac{1}{s}|f|\right) \leqq 1 \right\}.$$

A formulation of the Δ_2 condition is that

$$K(r) = \sup_{u \geqq r} \frac{up(u)}{M(u)} < \infty \qquad \text{for some } r > 0.$$

Of course this is the case if and only if $K(r) < \infty$ for all $r > 0$. Note that $\sup_{r>0} K(r)$ need not be finite. If M satisfies Δ_2 then

$$M(\lambda u) \leqq \lambda^{K(r)} M(u) \qquad \text{whenever } 0 < r \leqq u \quad \text{and} \quad 1 \leqq \lambda.$$

Hence, in this case, $\int M(\lambda |f|) < \infty$ for some $\lambda > 0$ if and only if $\int M(\lambda |f|) < \infty$ for all $\lambda > 0$. If Δ_2 is satisfied then it is easy to see that $\int M(\lambda |f|)$ is a continuous function of λ. Hence, in particular,

$$\int M\left(\frac{1}{\|f\|}|f|\right) = 1 \qquad \text{for all } f \in L_M.$$

We now note the following uniform continuity properties of this function.

2.10. LEMMA. *If Δ_2 is satisfied then for each $\epsilon > 0$ there is a $\delta > 0$ such that if $f \geqq 0$ with $\int M(f) \leqq 1$ then $\int M((1+\delta)f) \leqq 1 + \epsilon$.*

PROOF. First choose $r > 0$ such that $M(2r)\mu(X) < \frac{1}{2}\epsilon$. Then choose $\delta, 0 < \delta < 1$ such that $(1+\delta)^{K(r)} < 1 + \frac{1}{2}\epsilon$. Hence if $A = \{x \mid f(x) \geqq r\}$, $B = \{x \mid f(x) < r\}$ and $1 \leqq \lambda \leqq 1 + \delta$ then

$$\int M(\lambda f) = \int_A M(\lambda f) + \int_B M(\lambda f) \leqq \lambda^{K(r)} \int_A M(f) + M(2r)\mu(X)$$

$$\leqq (1+\delta)^{K(r)} + \frac{\epsilon}{2} \leqq 1 + \epsilon.$$

2.11. LEMMA. *If Δ_2 is satisfied then for each $\lambda \geqq 1$ there is an $\epsilon > 0$ such that if $f \geqq 0$ with $\int M(\lambda f) \geqq 1$ then $\int M(f) \geqq \epsilon$.*

PROOF. Choose $r > 0$ such that $M(\lambda r)\mu(X) \leqq \frac{1}{2}$. Then, defining A and B as in the previous proof, we have

$$1 \leqq \int M(\lambda f) = \int_A M(\lambda f) + \int_B M(\lambda f) \leqq \lambda^{K(r)} \int_A M(f) + \frac{1}{2}$$

which shows that

$$\epsilon = \tfrac{1}{2}\lambda^{-K(r)} \leqq \int M(f).$$

2.12. LEMMA. *If M satisfies Δ_2 then L_M satisfies* (C) *in the following uniform form: For each $\alpha > 0$ there is a $\beta > 0$ such that $\|f + g\| \geqq \|g\| + \beta$ whenever $f, g \in L_M^+$ with $\|f\| > \alpha$, $\|g\| \leqq 1$. If M does not satisfy Δ_2 then L_M does not satisfy* (C).

PROOF. Assume that Δ_2 is satisfied. Let $\alpha > 0$ be given. We assume, without loss of generality, that $\alpha \leqq 1$. Let $f, g \in L_M^+$, $\|f\| > \alpha$, $\|g\| \leqq 1$. First assume $\|g\| = 1$. Hence

$$\int M\left(\frac{1}{\alpha} f\right) \geqq 1, \qquad \int M(g) = 1.$$

Find $\epsilon > 0$ from Lemma 2.11, corresponding to $\lambda = 1/\alpha$. Then $\int M(f) \geqq \epsilon$ and

$$\int M(f+g) \geqq \int M(f) + \int M(g) \geqq 1 + \epsilon.$$

Then Lemma 2.10 shows that there is a $\beta_1 > 0$, depending only on $\epsilon > 0$

(consequently depending only on α), such that

$$\int M\left(\frac{f+g}{1+\beta_1}\right) > 1.$$

Hence $\|f+g\| \geqq 1+\beta_1$. Note that $0 < \beta_1 \leqq \alpha \leqq 1$. Let $\beta = \frac{1}{2}\alpha\beta_1$. We claim that $\|f+g\| \geqq \|g\| + \beta$ whenever $f, g \in L_M^+$, $\|f\| > \alpha$ $\|g\| \leqq 1$. In fact, if $\|g\| < \alpha/2$ then

$$\|f+g\| \geqq \alpha \geqq \|g\| + \frac{\alpha}{2} \geqq \|g\| + \beta.$$

If $\alpha/2 \leqq s = \|g\| \leqq 1$, then

$$\left\|\frac{1}{s}f + \frac{1}{s}g\right\| \geqq 1 + \beta_1$$

shows that $\|f+g\| \geqq s + s\beta_1 \geqq \|g\| + \beta$.

Now assume that Δ_2 is not satisfied. Then we will show that there are two functions $f, g \in L_M^+$ such that $\|f\| = \|g\| = \|f+g\| = 1$, which clearly violates (C). In fact, first find a sequence $u_n > 0$ such that

$$M(u_n) \geqq 1 \quad \text{and} \quad \frac{u_n p(u_n)}{M(u_n)} \geqq n \qquad \text{for all } n = 1, 2, \ldots.$$

Then choose another sequence $\alpha_n > 0$ such that $\sum_{n=1}^{\infty} \alpha_n = \frac{1}{2}$, $\sum_{n=1}^{\infty} n\alpha_n = \infty$. Assuming $\mu(X) \geqq 1$, let A_n, B_n be a family of pairwise disjoint sets such that

$$\mu(A_n) = \mu(B_n) = \frac{\alpha_n}{M(u_n)}.$$

Finally let

$$f = \sum_{n=1}^{\infty} u_n \chi_{A_n}, \qquad g = \sum_{n=1}^{\infty} u_n \chi_{B_n}.$$

Then it is clear that $\int M(f) = \int M(g) = \frac{1}{2}$ and $\int M(f+g) = \int M(f) + \int M(g) = 1$. Since

$$M((1+\epsilon)u_n) \geqq M(u_n) + \epsilon u_n p(u_n) \geqq (1+\epsilon n) M(u_n) \qquad \text{for each } \epsilon > 0,$$

we also see that $\int M((1+\epsilon)f) = \int M((1+\epsilon)g) = \infty$ whenever $\epsilon > 0$. Hence $\|f\| = \|g\| = \|f+g\| = 1$.

2.13. Remark. If $\mu(X) = \infty$ (but still non-atomic) then L_M satisfies (C) if and only if M satisfies Δ_2 in a stronger form: $K = \sup_{r>0} K(r) < \infty$. We omit the easy proof. We also note that in Orlicz spaces (C) is satisfied if and only if (C) is satisfied in a uniform form: For each $\alpha > 0$ there is a $\beta > 0$ such that $\|f\| \geqq \alpha$, $\|g\| \leqq 1$, $f, g \geqq 0$ implies $\|f+g\| \geqq \|g\| + \beta$. This is not the case in a general Banach lattice, as the following example shows. Let L consist of functions $f : [1, \infty) \to \mathbf{R}$ with

$$\|f\| = \sum_{n=1}^{\infty} \left[\int_n^{n+1} |f(x)|^n dx \right]^{1/n}.$$

It is easy to see that (C) is satisfied but not uniformly.

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